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SCINTILLATIONS AND LÉVY FLIGHTS THROUGH THE INTERSTELLAR MEDIUM

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ABSTRACT

Temporal broadening of pulsar signals results from electron density fluctuations in the interstellar medium that cause the radiation to travel along paths of different lengths. The theory of Gaussian fluctuations predicts that the pulse temporal broadening should scale with the wavelength as λ^4 and with the dispersion measure (DM; proportional to the distance to the pulsar) as DM^2. However, for large dispersion measures, $\mathrm{DM} > 20$ pc cm^{-3}, the observed scaling is $\lambda^4\mathrm{DM}^4$, contradicting the conventional theory. Although the problem has existed for 30 years, there has been no resolution to this paradox. We suggest that scintillations for distant pulsars are caused by non-Gaussian, spatially intermittent density fluctuations with a power-law–like probability distribution. Such a probability distribution does not have a second moment, and therefore the previously applied conventional Fokker-Planck theory does not hold. Instead, we propose to apply the theory of Lévy distributions (so-called Lévy flights). We show that the observed scaling is recovered for large DM if the density differences, ΔN, have Lévy distribution decaying as $|\Delta N|^{-5/3}$. In the thin-screen approximation, the corresponding tail of the time-profile of the arriving signal is estimated to be $I(\tau) \propto \tau^{-4/3}$.

Subject headings: ISM: kinematics and dynamics — ISM: structure — turbulence

1. INTRODUCTION

Intensity fluctuations of pulsars' radiation result from the scattering of radio waves by electron-density inhomogeneities in the interstellar medium (ISM). These fluctuations are a signature of turbulent, nonequilibrium motion in the ISM, and as the phenomenon of turbulence itself, they have withstood full theoretical understanding for decades, see, e.g., reviews by Sutton (1971) and Rickett (1977, 1990). Observationally, the presence of electron density fluctuations leads, among other effects, to temporal and angular broadening of the pulsar image. These two effects are naturally related—because of fluctuations of the refraction index, different rays from a pulsar travel along paths of different shapes and the larger the deviation of the path from the straight line, the broader the pulsar image and the greater the time broadening of the arriving signal. Denoting the angular width of the image as $\Delta\theta$, and using simple geometric considerations, one estimates the arrival time broadening as $\tau_d \sim (\Delta\theta)^2 d/c$, where d is the distance to the pulsar and c is the speed of light; see a more detailed discussion in Blanford & Narayan (1985) and Gwinn, Bartel, & Cordes (1993).

A ray propagating through the ISM encounters many randomly distributed small " prisms " on its way that make the scattering angle wander randomly. At each scattering event, the angle deflection is proportional to λ^2 (see below), where λ is the wavelength of the scattered radiation. Taking into account that the scattering angle is small and exhibits a standard Gaussian random walk, we estimate that $(\Delta\theta)^2 \propto \lambda^4 d$ and that the time delay scales as $\tau_d \propto \lambda^4 d^2$, where d is proportional to the number of steps in this random walk.

The distance to the pulsar is approximately proportional to the observable dispersion measure, DM, and therefore this relation can be checked experimentally. As has been consistently noted for more than 30 years, observed scaling of scintillations of distant pulsars, $\mathrm{DM} > 20$ pc cm^{-3}, is far from this simple theoretical prediction; instead, it is well described by $\tau_d \propto \lambda^4\mathrm{DM}^4$ (see, e.g., Sutton 1971; Rickett 1977). Sutton proposed that this scaling for longer lines of sight arose from the gradually increasing probability of intersection with more strongly scattering H II regions. In this sense, he proposed that rare, large events dominated the line-of-sight averages, via nonstationarity in the statistics of electron density.

The problem of scintillations has been addressed by many authors who developed thorough analytical models (see the discussion in Tatarskii & Zavorotnyi 1980; Rumsey 1975; Gochelashvily & Shishov 1975; Lee & Jokipii 1975a, 1975b, 1975c; Goodman & Narayan 1985; Blanford & Narayan 1985; Lithwick & Goldreich 2001). These theories all assume that the density difference between two points is drawn from a Gaussian distribution, with the variance of the distribution being a function of the separation of the points. They characterize the variation of density with position by the projected correlator of density fluctuations. This statistic gives the second moment of the difference in projected densities.

Let the electron density be denoted as $N(\boldsymbol{r})$ and its projection perpendicular to the distance d as $\tilde{N}(\boldsymbol{x}) = \int_0^d dz\, N(\boldsymbol{r})$. Here $\boldsymbol{x}$ is a two-dimensional vector in the plane perpendicular to the line of sight, and z is a coordinate along the line of sight, i.e., $\boldsymbol{r} = (\boldsymbol{x}, z)$. Note that these theories all assume that the distribution of projected density fluctuations is Gaussian and that it is completely described by its second

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moment, the projected density correlator. The density and projected density correlators are related as

$$\langle \tilde{N}(\boldsymbol{x}_1)\tilde{N}(\boldsymbol{x}_2)\rangle = \int_0^d \int_0^d dz_1\, dz_2\, \langle N(\boldsymbol{r}_1)N(\boldsymbol{r}_2)\rangle \ , \quad (1)$$

where both fields inside the brackets are taken at the same time. Because of space homogeneity, these correlators depend only on the difference of the coordinates, e.g., $\langle N(\boldsymbol{r}_1)N(\boldsymbol{r}_2)\rangle = \kappa(\boldsymbol{r}_1 - \boldsymbol{r}_2)$. The special case in which $\kappa(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ is Gaussian is sometimes called a "Gaussian spectrum of density fluctuations"; this special case is quite distinct from the standard assumption that the distribution of density differences between two given points is Gaussian. In this paper we relax this standard assumption and investigate power-law–like distributions of density-difference fluctuations at two given points.

Assuming that the density fluctuations have finite correlation length l, i.e., that the κ function decays fast for $|\boldsymbol{r}_1 - \boldsymbol{r}_2| > l$, we obtain

$$\langle \tilde{N}(\boldsymbol{x}_1)\tilde{N}(\boldsymbol{x}_2)\rangle = d\int_0^\infty dz\, \kappa(\boldsymbol{x}_1 - \boldsymbol{x}_2, z) \equiv d\, \tilde{\kappa}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \ , \quad (2)$$

where $z = z_1 - z_2$, and we assume $d \gg l$. It is easy to show that if in the inertial range of turbulent fluctuations, $|r| \ll l$, the κ function behaves as $\kappa(\boldsymbol{r}) \sim N_0^2(1 - B(r/l)^\alpha + \ldots)$, then the projected function is expanded as $\tilde{\kappa}(x, t) \sim \tilde{N}_0^2(1 - \tilde{B}(x/l)^{1+\alpha} + \ldots)$. As an estimate, one has $\langle N^2\rangle = N_0^2 \sim \tilde{N}_0^2/l$, and B and $\tilde{B}$ are of the order 1. The analytical (smooth) case corresponds to $\alpha = 1$, and in this case $\tau_d \propto \lambda^4 d^2$. In general, the density field need not be analytic, and $\alpha \neq 1$. This may happen when the density fluctuations arise as a result of a turbulent cascade. For example, Kolmogorov turbulence would imply $\alpha = \frac{2}{3}$. Rigorous consideration shows that in the nonanalytic case, the scaling of the broadening time changes. Various possibilities have been exhaustively analyzed in the literature (see, e.g., Lee & Jokipii 1975a, 1975b; Goodman & Narayan 1985; Lambert & Rickett 2000). For $\alpha \leq 1$, one obtains

$$\tau_d \propto \lambda^{2(\alpha+3)/(\alpha+1)} d^{(\alpha+3)/(\alpha+1)} \ , \quad (3)$$

while for a more exotic case, $\alpha > 1$, one gets

$$\tau_d \propto \lambda^{8/(3-\alpha)} d^{(3+\alpha)/(3-\alpha)} \ . \quad (4)$$

In § 2 we present a simple derivation of these results. Since most observational data indicate that λ-scaling is close to λ^4, neither possibility provides enough freedom for changing the d-scaling from d^2 to d^4.

In the present paper we propose a new model that fully exploits the turbulent origin of the density fluctuations. We assume that the statistics of the density fluctuations are not Gaussian, but highly intermittent, and that the probability density function (PDF) of density differences decays as a power law, $P(\Delta N) \propto |\Delta N|^{-1-\beta}$. If this power-law distribution has a *divergent* second moment ($\beta < 2$), the Gaussian random walk approach does not work. Instead, we suggest the use of the theory of Lévy distributions (see Shlesinger, Zaslavsky, & Frisch 1995). Physically, the possibility of power-law density distribution seems rather natural for strong turbulent fluctuations. Indeed, the ISM turbulence can be near-sonic, i.e., velocity and density fields can develop shock discontinuities. From the theory of shock turbulence (Burgers turbulence) one knows that large negative velocity gradients or shocks have a power-law distribution (Polyakov 1995; E et al 1997; Boldyrev 1998). Jump conditions on a shock then show that the velocity and density discontinuities are proportional to each other, and so density jumps may also have a power-law distribution. We note that this explanation is related to that proposed by Sutton (1971) in that rare, large events play a dominant role in scattering; however, we assume that the statistics *are* stationary.

Taking the Lévy distribution of the density fluctuations as a working conjecture, we demonstrate that the scaling of the broadening time with respect to d is sensitive to the exponent of the distribution, β, and the scaling $\tau_d \propto \lambda^4 d^4$ is reproduced for $\beta = \frac{2}{3}$.

In the next section we review the ray-tracing model of pulse propagation, considered previously by Williamson (1972, 1973) and Blanford & Narayan (1985). In particular, we rederive the results cited above for Gaussian density fluctuations in a general, nonanalytic case. In § 3 we apply the model to the non-Gaussian, Lévy-distributed density fluctuations. We then numerically calculate the distribution of pulse-arrival times in the case of a smooth density field and demonstrate that if $P(\Delta N) \propto |\Delta N|^{-5/3}$, the width of this distribution changes with the distance to the pulsar as $\lambda^4 d^4$, in agreement with our scaling arguments, and the tail of the distribution has a power-law decay. Conclusions and future research are outlined in § 4.

2. RAY-TRACING METHOD

Ray-tracing is applicable in the limit of geometric optics, i.e., when the wave length is much smaller than the characteristic size of density inhomogeneities (Lifshitz, Landau, & Pitaevsky 1995). This rather effective method was applied to the problem of scintillations by Williamson (1972, 1973) and Blanford & Narayan (1985); we present it here in the form that allows us to apply it in the next section to Lévy walks. In the limit considered, signal propagation can be characterized by rays, $\boldsymbol{r}(t)$, along which wave packets travel similar to particles obeying the following system of Hamilton equations:

$$\dot{\boldsymbol{r}} = \partial\omega(k, r)/\partial\boldsymbol{k} \ ,$$
$$\dot{\boldsymbol{k}} = -\,\partial\omega(k, r)/\partial\boldsymbol{r} \ . \quad (5)$$

In this representation, ω plays the role of the Hamiltonian, $\omega^2 = \omega_{\rm pe}^2(r) + k^2c^2$, where $\omega_{\rm pe}^2(r) = 4\pi N(r)e^2/m_e$ is the electron plasma frequency and $\boldsymbol{k}$ is a characteristic wavevector of the packet. Differentiating the first expression in equation (5) with respect to t and using the second expression in equation (5) one obtains

$$\ddot{\boldsymbol{r}} = -2\pi c^2\lambda^2 r_0 \partial N(r)/\partial\boldsymbol{r} \ , \quad (6)$$

where $r_0 = e^2/m_ec^2$ is the classical radius of electron. Taking into account that the ray propagates at small angles to the line of sight, chosen as the z-axis, we are interested in the ray displacement in the perpendicular, $\boldsymbol{x}$ direction, and instead of time t we use the variable $z = ct$. (We assume that $\omega \gg \omega_{\rm pe}$, a condition well satisfied for interstellar propagation). The ray trajectory can now be viewed as the function $\boldsymbol{x}(z)$. Consider now two rays, separated by a vector

$\Delta \boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ in the direction perpendicular to the z-axis. As follows from equation (6), this vector obeys the equation

$$\frac{d(\Delta \boldsymbol{x})}{dz} = \Delta\theta\,,$$
$$\frac{d(\Delta\theta)}{dz} = A\left[\frac{\partial N(x_1, z)}{\partial \boldsymbol{x}_1} - \frac{\partial N(x_2, z)}{\partial \boldsymbol{x}_2}\right] \equiv A\Delta\frac{\partial N}{\partial \boldsymbol{x}}(z)\,, \quad (7)$$

where $A = -2\pi\lambda^2 r_0$, and $\Delta\theta(z)$ is, generally, of the order of the angle between the directions of the rays.

Let us now *assume* that the electron-density difference is a Gaussian random function with correlation length l. Then $\Delta\theta(z)$ is a Gaussian random walk, whose elementary step has the duration l in z direction. Since we are interested in very large propagation distances, $z \gg l$, and the scattering angles are very small, one can effectively assume that the random density is short-range correlated, i.e., the characteristic "z-scale" of change of vectors $\Delta\theta$ and Δx is much larger than l (Markov approximation). In principle, our technique describes propagation from a point source to a point observer through a statistically homogeneous medium, although it can easily be generalized. We extract only scaling laws below, so our results are not sensitive to this assumption.

Formally, the assumption of short-range correlation of density field means that $\kappa(\boldsymbol{r}_1 - \boldsymbol{r}_2) = 2\tilde{\kappa}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\delta(z_1 - z_2)$. Equation (7) is thus completely analogous to the Langevin equation for a particle having coordinate Δx and velocity $\Delta\theta$. The diffusion coefficient for the particle motion is given by a standard formula,

$$D = A^2\int_0^\infty \left\langle \Delta\frac{\partial N}{\partial \boldsymbol{x}}(z_1)\Delta\frac{\partial N}{\partial \boldsymbol{x}}(z_2)\right\rangle d(z_1 - z_2)$$
$$\sim \lambda^4 r_0^2 N_0^2 \left(\frac{\Delta x}{l}\right)^{\alpha-1}\frac{1}{l}\,, \quad (8)$$

where the diffusion is described by $(\Delta\theta)^2 \sim Dz$. We observe however that the diffusion coefficient depends on the distance Δx, and its behavior differs qualitatively for $\alpha < 1$ and $\alpha > 1$. In the first case, $\alpha < 1$, diffusion is larger for smaller distances, and therefore two rays effectively attract each other in the course of propagation. This means that our geometric ray picture breaks down, and one needs to consider the effects of interference (interaction) of different rays. This happens when the beam is compressed to the size limited by the uncertainty condition in the perpendicular direction , $k\Delta\theta\Delta x \sim 1$. Upon substituting $\Delta\theta \sim D^{1/2}z^{1/2}$ and using the expression for the diffusion coefficient (eq. [8]), we obtain the minimal size of contraction and, equivalently, the diffraction angle corresponding to an aperture of this size. Assuming that the contraction happens at about half the distance between the pulsar and the Earth, $z \sim d/2$, we find

$$(\Delta\theta)^2 \sim \left[N_0^4 r_0^4 l^{-2\alpha}\lambda^{2(\alpha+3)}d^2\right]^{1/(\alpha+1)}\,, \quad \alpha < 1\,. \quad (9)$$

Recalling now that $\tau_d \sim (\Delta\theta)^2 d/c$, we recover the result given by equation (3). A rigorous wave analysis gives essentially the same result as our "semiclassical" approach. In the second case, $\alpha > 1$, the rays effectively repel, so geometric optics do not break down. In this case $\Delta x \sim \Delta\theta z \sim D^{1/2}z^{3/2}$. This equation gives

$$(\Delta\theta)^2 \sim \left(N_0^4 r_0^4 l^{-2\alpha}\lambda^8 d^{2\alpha}\right)^{1/(3-\alpha)}\,, \quad 1 \le \alpha < 3\,, \quad (10)$$

which agrees with the result of equation (4). Both expressions give the same result for the analytic case, $\alpha = 1$. The above standard results have been obtained by many authors and by a variety of different methods (see, e.g., Williamson 1972; Lee & Jokipii 1975a, 1975b; Goodman & Narayan 1985; Blanford & Narayan 1985). As we mentioned in the introduction, neither of the expressions (eqs. [9] or [10]) allows us to recover the observed scaling $\tau_d \propto \lambda^4 d^4$. In the next section we address the problem, assuming that the density-difference distribution has a slowly decaying power-law tail, such that the second moment of the distribution does not exist. In this case the diffusion approximation does not hold, and one needs to work directly with equation (7) to establish the scaling of the probability of pulse arrival times.

3. LÉVY MODEL FOR SCINTILLATIONS

In previous sections we implicitly used the central limit theorem, which states that the sum of many independent random variables (in our case, angle deflections) has a Gaussian distribution, if second moments of these variables exist. More precisely, a convolution of many distribution functions that have second moments, converges to an appropriately scaled Gaussian distribution. Therefore, the convolution of two Gaussian functions is again a rescaled Gaussian function. One can generalize this question for distribution functions without finite second moments: if their convolution converges, what is the limit? The answer is the so-called Lévy distribution (Shlesinger et al 1995). As is the Gaussian distribution, the Lévy distribution is stable: convolution of this distribution with itself gives the same distribution after proper rescaling. In other words, if two independent random variables are drawn from a Lévy distribution, their sum has the same distribution, appropriately rescaled. Analogously to a Gaussian random walk, a sum of independent, Lévy-distributed random variables is called a Lévy walk or Lévy flight. The latter name reflects the highly intermittent behavior of a typical Lévy trajectory: it has sudden large jumps or "flights;" see Figures 1 and 2. Lévy

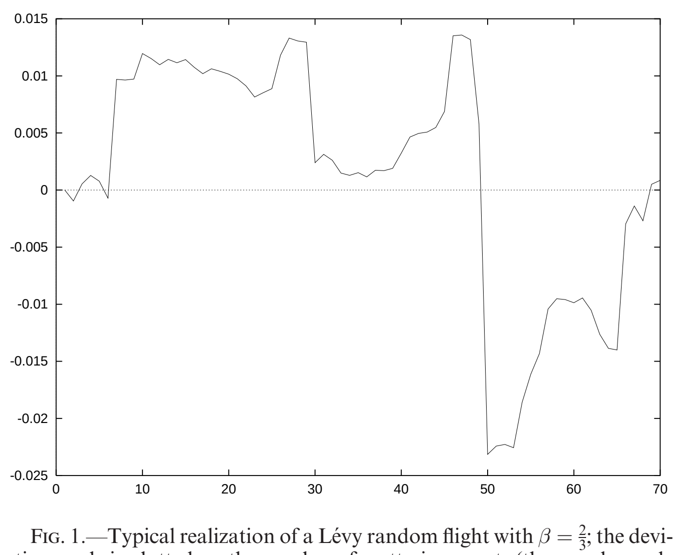

FIG. 1.—Typical realization of a Lévy random flight with $\beta = \frac{2}{3}$; the deviation angle is plotted vs. the number of scattering events (the angular scale is arbitrary). The trajectory exhibits sudden large deviations. In the case of ray propagation through the ISM, the ray angle performs a Lévy flight. Large angular deviations occur when the ray encounters regions of large electron density inhomogeneities, such as shocks or H II regions.

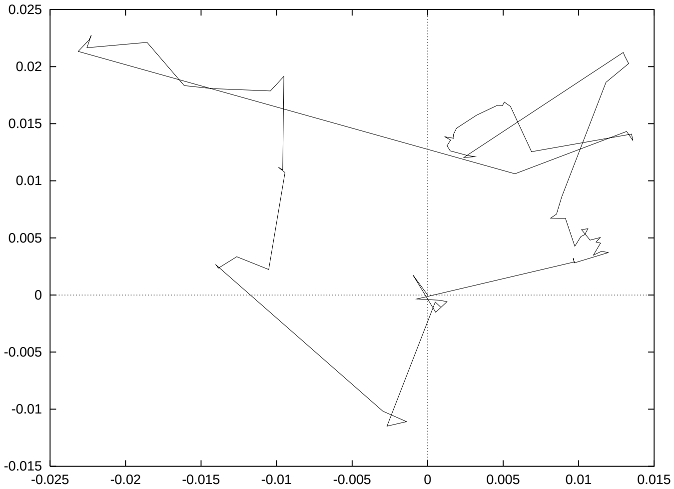

FIG. 2.—Typical realization of a two-dimensional Lévy flight with $\beta = \frac{2}{3}$. The axes show ray angle deviations in x and y directions (the angular scale is arbitrary).

flights are common in completely different random systems and often replace diffusion in turbulent systems. For example, a particle exhibiting a Brownian random motion in an equilibrium fluid, exhibits a Lévy walk in a turbulent fluid. For a variety of further illustrations see Shlesinger et al. (1995).

If a random variable y has a symmetric Lévy probability density, $P(y)$, then the Fourier transform of this distribution (the characteristic function) has the form

$$\Phi(\mu) = \int_{-\infty}^{\infty} dy\, P(y) \exp(i\mu y) = \exp(-C|\mu|^{\beta}) , \quad (11)$$

where $0 < \beta < 2$ and C is some positive constant. Equation (11) can be taken as the definition of a symmetric Lévy walk. In the special case $\beta = 2$ we recover a Gaussian distribution. For $0 < \beta < 2$ one can verify that $P(y) \propto |y|^{-1-\beta}$ as $|y| \to \infty$, and the second moment of this distribution diverges. Of course, a distribution of a physical quantity usually has a second moment. This does not contradict our case, since the far tails of the PDF, which are not described by the Lévy equation (11), make the dominant contribution to the second moment. However, if we are interested in effects caused by small fluctuations, $y \ll y_{\rm rms}$, it is the central part of the PDF that is important.

The characteristic function of a convolution of n Lévy distributions is just a product of n characteristic functions (eq. [11]). We therefore conclude that the sum of n Lévy distributed random variables has the distribution

$$P_n(y) = P(yn^{-1/\beta})n^{-1/\beta} . \quad (12)$$

This is the demonstration of the convolution *stability* of the Lévy distribution. Formula (12) teaches us that the displacement y of the Lévy random walk scales with the number of steps as $y \sim n^{1/\beta}$. In the Gaussian case, $\beta = 2$, we recover the well known diffusion result.

We now apply this result to our scintillation problem. Let us *assume* that the dimensionless density difference $\Delta N(x)/N_0$ has a Lévy distribution with parameter β. We then obtain from equation (7), $\Delta\theta \sim -A\Delta N$, and (compare this result to eq. [8])

$$(\Delta\theta)^2 \sim \lambda^4 r_0^2 N_0^2 \left(\frac{\Delta x}{l}\right)^{\alpha-1} \left(\frac{z}{l}\right)^{2/\beta} . \quad (13)$$

In this formula, α describes the scaling of the density fluctuations with separation, while β is the exponent of the power-law decay of the probability distribution function for density-difference fluctuations at a given separation. Note that the scaling in equation (13) is understood, not in the sense of averaging, but in the sense of scaling of the central part of the distribution, $P_z(\Delta\theta)$. For example, the scaling with distance z is understood as $P_{zn}(\Delta\theta n^{1/\beta})n^{1/\beta} = P_z(\Delta\theta)$, for any n (see eq. [12]). This scaling behavior holds for the bulk of scattering angles, as shown by the typical widths of pulses in Figure 3. The extreme tails of the distribution function of $\Delta\theta$, which are not described by equation (11), dominate the moments $\langle(\Delta\theta)^m\rangle$ for $m > \beta$. In the Lévy model (eq. [11]), these moments do not exist. In practice, at extremely large $\Delta\theta$, we expect the scattering law to depart from the predictions of this model. Indeed, the moments may show characteristic scaling behavior, quite possibly different from that described in equation (13). We expect that the effects from this scaling might be important for extremely large time delays τ. We therefore wish to caution the reader that the scaling exponent α that we use in this section can be different from the second-moment exponent α, introduced after equation (2).

We now proceed exactly as we did in the derivation of equations (9) and (10), and obtain for $\alpha < 1$,

$$(\Delta\theta)^2 \sim \left[N_0^4 r_0^4 l^{2-2\alpha-4/\beta} \lambda^{2(\alpha+3)} d^{4/\beta}\right]^{1/(\alpha+1)} , \quad (14)$$

and for $1 \le \alpha < 3$,

$$(\Delta\theta)^2 \sim \left[N_0^4 r_0^4 l^{2-2\alpha-4/\beta} \lambda^8 d^{2\alpha-2+4/\beta}\right]^{1/(3-\alpha)} . \quad (15)$$

In the smooth (analytic) case, $\alpha = 1$, the scaling of the time

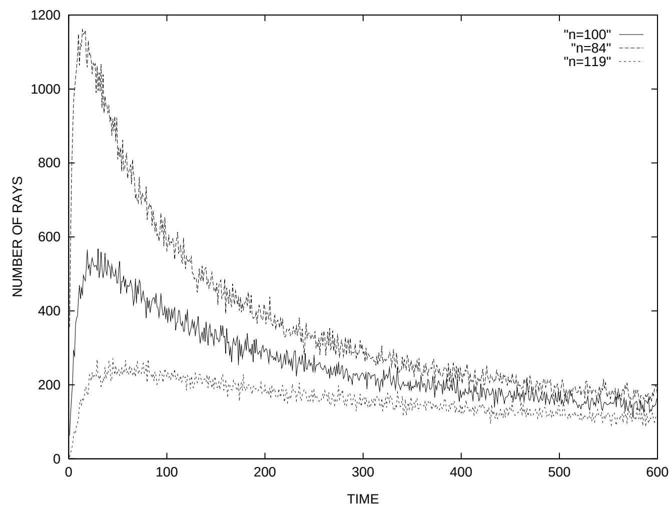

FIG. 3.—Numerical calculation of the number of arriving rays vs. time (time units are arbitrary). We used eq. (17) and the Lévy-distributed density fluctuations with $\beta = \frac{2}{3}$. We calculated arrival times of 10^6 rays for three different distances to the source, $n_1 = 100$, $n_2 = 84 \approx 100 \times 2^{-1/4}$, and $n_3 = 119 \approx 100 \times 2^{1/4}$. The width of the plot $n = 84$ is twice as small, and the width of the plot $n = 119$ is twice as large, as the width of the plot $n = 100$. This corresponds to the scaling $\tau_d \propto d^4$.

broadening is

$$\tau_d \sim (N_0^2 r_0^2 l^{-2/\beta}/c)\lambda^4 d^{(2+\beta)/\beta} . \quad (16)$$

We see that this scaling is sensitive to the exponent of the power distribution of the density fluctuations, and $\tau_d \propto \lambda^4 d^4$ is achieved for $\beta = \frac{2}{3}$. This result was obtained by rather general arguments and describes the scaling of the arrival time distribution, rather than the moments of this distribution. Observations measure precisely the time width of the arriving signal, not its moments; i.e., they infer exactly the quantity corresponding to scaling equation (16).

In the rest of this section we would like to verify the scaling equation (16) by numerical simulation of equation (7), and to get some idea about the time shape of the arriving signal. Let us assume that the distance to the pulsar, d, is much larger than the scale of an elementary scatter, l, i.e., $n = d/l \gg 1$, where n is the number of scattering events. We also assume that $\alpha = 1$. At each scattering event, the angle of the ray changes by $\delta\theta \ll 1$, where $\delta\theta$ is a Lévy-distributed random variable. (We designate δN and $\delta\theta$ the characteristic changes of the density field and of the angle deviation of a ray, introduced by one scattering segment of length l along the line of sight. This should not be confused with the changes of these variables between two *different* rays in a perpendicular plane $\boldsymbol{x}$, denoted by Δ.) The time delay (compared to the straight propagation) introduced by each scattering segment is $\delta\tau_d = l\theta^2/(2c)$. We want to find the probability distribution of the total travel time delay,

$$\tau_d = \frac{l}{2c}\sum_{m=1}^{n}\theta_m^2 = \frac{l}{2c}\sum_{m=1}^{n}\left(\sum_{s=1}^{m}\delta\theta_s\right)^2 , \quad (17)$$

assuming that each $\delta\theta_s$ (where $\delta\theta_s \sim -A\delta N$ as a consequence of eq. [7]) is distributed identically, independently, and according to the Lévy law (eq. [11]) with $\beta = \frac{2}{3}$.

In Figure 3 we plot the number of arriving rays as a function of time. We considered the number of scattering events (the distance to the pulsar) to be $n_1 = 100$, $n_2 = 119 \approx 100 \times 2^{1/4}$, and $n_3 = 84 \approx 100 \times 2^{-1/4}$. From Figure 3 one can see that the widths of the curves (estimated at half of their maximum values) indeed differ by a factor of 2, as the scaling $\tau_d \propto \lambda^4 d^4$ would predict for these distances. The curves obtained closely resemble the real-time shapes of arriving signals, although they cannot be trusted for small propagation angles (small times) since the model does not capture effects of interference. The tails of the signals are however reliably predicted to have an asymptotic power-law form. This asymptotic behavior can be found from the following qualitative argument (which would be exact for a thin screen model and for $\alpha = 1$). The delay time is proportional to the square of the typical deviation angle of the trajectory, $\tau \propto \theta^2$, where θ has a Lévy distribution $P(\theta)$ with $\beta = \frac{2}{3}$. Therefore, the distribution of arrival times can be found as $I(\tau) \propto P(\tau^{1/2})\tau^{-1/2}$, having the asymptotic form $I(\tau) \propto \tau^{-4/3}$ as $\tau \to \infty$. In contrast, the traditional thin-screen model with a Kolmogorov spectrum yields an asymptotic form $I(\tau) \propto \tau^{-11/6}$ (Isaacman & Rankin 1977). The curves presented in Figure 3 have an asymptotic form close to $I(\tau) \propto \tau^{-1.2}$, which is not far from our qualitative estimate.

4. CONCLUSIONS

We suggest a novel explanation for the observed scaling of time broadening of pulsar signals for large distances (large dispersion measures, DM > 20 pc cm^{-3}), $\tau_d \propto \lambda^4 d^4$. The central concept is that the density fluctuations in the interstellar medium have a Lévy probability distribution function that has power-law decay and has no second moment. The angle of pulse propagation, deviated by these density fluctuations, exhibits not a conventional Brownian motion, but rather a Lévy flight. The exponent β is the parameter of the probability distribution of density differences, and the pulse broadening time is rather sensitive to it, as is described by our main equations (14) and (15). The observed scaling $\tau_d \propto \lambda^4 d^4$ is recovered for $\beta = \frac{2}{3}$, i.e., for the $|\Delta N|^{-5/3}$ decay of the distribution function of density differences.

Further investigation is needed to explain the proposed $-5/3$ exponent. We believe that observations have potentially great impact on theory in this case, because they can discriminate among different possible distribution functions for the density-difference fluctuations (measured at a fixed-point separation), as well as measuring the spatial spectrum of density fluctuations. This PDF appears as a result of turbulent density fragmentation, and it would be highly desirable to search for such fragmentation in numerical simulations or to develop an analytical explanation for it. Calculations of propagation including wave phenomena, such as diffraction and interference, may provide predictions for the precise shapes of scattered pulses, particularly at large delays, and averaged over long observing intervals, as a function of β. Comparison of temporal or angular broadening observations with these predictions may help understand interstellar turbulence. These represent concrete predictions of our model for the turbulence in the ISM, which can in principle be checked numerically or observationally.

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